

# Holography and $AdS_4$ self-gravitating dyons

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## Abstract

We present a self-gravitating dyon solution of the Einstein-Yang-Mills-Higgs equations of motion in asymptotically AdS space. The back reaction of gauge and Higgs fields on the space-time geometry leads to the metric of an asymptotically AdS black hole. Using the gauge/gravity correspondence we analyze relevant properties of the finite temperature quantum field theory defined on the boundary. In particular we identify an order operator, characterize a phase transition of the dual theory on the border and also compute the expectation value of the finite temperature Wilson loop.

## 1 Introduction

The AdS/CFT correspondence [1]-[3] originally proposed to connect gauge and string theories has recently became a powerful tool to study strong coupling physics in systems of interest in quantum field theory and condensed matter physics from the analysis of the low energy approximation of

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string theories, namely gravitational theories [4]-[7] (for references on recent progress see for example [8]).

In a previous work [9] we have used such gauge-gravity duality to relate a  $d = 3+1$  Yang-Mills Higgs model in a Schwarzschild-Anti-de Sitter black hole background with a  $d = 2 + 1$  quantum field theory defined on the boundary. In this way we were able to study the strong coupling regime of the theory on the boundary from the classical dyon solution found in the bulk, showing that the  $2 + 1$  system undergoes a second order phase transition and exhibits the condensation of a composite charge operator.

It is the purpose of the present work to extend our study to the case in which there is back reaction on the space-time geometry so that the complete Einstein-Yang-Mills-Higgs (EYMH) system of equations of motion has to be solved, with the condition that the metric solution corresponds to a black hole in asymptotically AdS space.

Self-gravitating dyon solutions of the EYMH equations in asymptotically AdS space, regular both at the origin and at infinity, were constructed in [10]-[11]. It was found that, as the value of the dimensionless parameter  $G_N H_0^2$  grows, the metric function develops a minimum which approaches to zero at a critical value ( $G_N$  is the Newton constant and  $H_0$  the v.e.v. of the Higgs scalar). Above the critical value a singularity at the origin develops indicating the possibility of a threshold horizon. In fact, black hole solutions that are finite at spatial infinity and at one horizon have been shown to exist in systems like the one we are interested for the case of asymptotically flat space [12]-[15]. One of the purposes of the present work is precisely to construct such black hole solutions in asymptotically  $AdS_4$  space, having finite mass and non-zero horizon radius. Since the black hole naturally introduces a temperature we shall be able to analyze relevant properties of the finite temperature quantum field theory defined on the boundary using the gauge/gravity correspondence.

It is interesting to notice that, in the context of the AdS/CFT correspondence, gauge fields in the  $AdS_d$  bulk can induce a field theory on the boundary containing dynamical gauge fields for low dimensions ( $d \leq 4$ ) [16]-[20]. It then becomes of interest to compute the expectation value of the finite temperature Wilson loop of the theory at the boundary which, according to the gauge/gravity correspondence [21]-[25], is related to the Nambu-Goto action in the bulk associated to the dyon solution.

The plan of the paper is the following. In section 2 we present the model

in the bulk  $\mathbf{M}$  and propose a spherically symmetric ansatz that reduces the EYMH equations of motion to a nonlinear system of coupled ordinary differential equations. Such ansatz corresponds to a  $\partial\mathbf{M} = \mathbf{S}^2 \times \mathbf{S}^1$  boundary (with the time direction compactified to  $\mathbf{S}^1$ ). We also consider the appropriate change of variables leading to  $\partial\mathbf{M} = \mathbf{R}^2 \times \mathbf{S}^1$  boundary which could be of interest in view of condensed matter applications. In section 3 we discuss the appropriate conditions on the fields and metric in order to have a solution giving a black hole in asymptotically  $AdS_4$  space and the gauge and Higgs fields of a dyon. We construct numerically such a solution and discuss its properties. We then apply the holographic correspondence to analyze the behavior of the system defined on the border as the temperature changes, identifying an order parameter and characterizing the phase transition that the  $2 + 1$  system undergoes. Section 4 is devoted to the calculation of the Wilson loop and binding energy for the theory in the border. Finally, we present a summary and discussion of our results in section 5.

## 2 The model

As in ref.[9], we consider a gravity-Yang-Mills-Higgs system with gauge group  $SU(2)$  and the scalar field in the adjoint representation, in a 4 dimensional space-time with Minkowski signature  $(-, +, +, +)$ . The action takes the form

$$S = S_G + S_{YM} + S_H = \int d^4x \sqrt{|g|} (L_G + L_{YM} + L_H) \quad (1)$$

with

$$L_G = \frac{1}{2\kappa^2} (R - 2\Lambda) \quad (2)$$

$$L_{YM} = -\frac{1}{4e^2} F_{\mu\nu}^a F^{a\mu\nu} \quad (3)$$

$$L_H = -\frac{1}{2} D_\mu H^a D^\mu H^a - \frac{\lambda}{4} (H^a H^a - H_0^2)^2 \quad (4)$$

Here  $\kappa^2 \equiv 8\pi G_N$  with  $G_N$  the Newton constant;  $e$  the gauge coupling and  $\Lambda$  the cosmological constant (we take  $\Lambda < 0$  which corresponds for our conventions and in the absence of matter, to anti-de Sitter space). For simplicity we will consider the BPS limit  $\lambda/e^2 = 0$ . The field strength  $F_{\mu\nu}^a$  ( $a = 1, 2, 3$ )

and the covariant derivative  $D_\mu$  acting on the Higgs triplet  $H^a$  are defined as

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \varepsilon^{abc} A_\mu^b A_\nu^c, \quad D_\mu H^a = \partial_\mu H^a + \varepsilon^{abc} A_\mu^b H^c \quad (5)$$

The most general static spherically symmetric form for the metric in 3 spatial dimensions together with the t'Hooft-Polyakov-Julia-Zee ansatz for the gauge and Higgs fields in the usual vector notation for internal degrees of freedom reads [10]-[11]

$$\begin{aligned} ds^2 &= -\mu(x) A(x)^2 dt^2 + \mu(x)^{-1} dr^2 + r^2 d\Omega_2 \\ \vec{A} &= dt e H_0 J(x) \check{e}_r - d\theta (1 - K(x)) \check{e}_\varphi + d\varphi (1 - K(x)) \sin \theta \check{e}_\theta \\ \vec{H} &= H_0 H(x) \check{e}_r \end{aligned} \quad (6)$$

where  $H_0$  sets the mass scale ( $[H_0] = m^1$ ) and we have introduced the dimensionless radial coordinate  $x \equiv e H_0 r$ .

## From $S^2$ to $R^2$

As stated in the introduction, we are interested in the physical system defined on the boundary  $\partial\mathbf{M}$  using the dual classical description of the gravity system governed by the action (1) in the 4 dimensional manifold  $\mathbf{M}$  which is, asymptotically, anti de Sitter space. This means that after compactifying in the time direction, one has  $\partial\mathbf{M} = \mathbf{S}^2 \times \mathbf{S}^1$ . Now, following the approach in [24] one can trade such boundary with  $\partial\mathbf{M} = \mathbf{R}^2 \times \mathbf{S}^1$  by introducing a dimensionless parameter  $R$  through a change coordinates  $(t, x, \theta, \varphi) \rightarrow (\tau, y, x^1, x^2)$  (or equivalently  $(t, x, \theta, \varphi) \rightarrow (\tau, y, \rho, \varphi)$ ), and then taking the  $R \rightarrow \infty$  limit. As we did in ref.[9], we consider the change of variables

$$\begin{aligned} \tau &= R t \\ y &= (\gamma_0/R) x = r/(LR) \\ x^1 &= 2 RL \tan(\theta/2) \cos \varphi = \rho \cos \varphi \\ x^2 &= 2 RL \tan(\theta/2) \sin \varphi = \rho \sin \varphi \end{aligned} \quad (7)$$

together with the field redefinitions

$$\begin{aligned} \tilde{A}(y) &= A(x), & \tilde{f}(y) &= R^{-2} \mu(x), \\ \tilde{H}(y) &= H(x), & \tilde{K}(y) &= R^{-1} K(x), & \tilde{J}(y) &= R^{-1} J(x) \end{aligned} \quad (8)$$

After this change, the ansatz (6) becomes

$$\begin{aligned}
ds^2 &= -\tilde{f}(y) \tilde{A}(y)^2 d\tau^2 + L^2 \frac{dy^2}{\tilde{f}(y)} + y^2 \frac{dx^{1^2} + dx^{2^2}}{(1 + \frac{\rho^2}{4(LR)^2})^2} \\
\vec{A} &= d\tau e H_0 \tilde{J}(y) \left( \frac{4LR\rho}{\rho^2 + 4(LR)^2} \check{e}_\rho + \frac{4(LR)^2 - \rho^2}{\rho^2 + 4(LR)^2} \check{e}_3 \right) \\
&\quad + \frac{4LR^2}{\rho^2 + 4(LR)^2} \left( \frac{1}{R} - \tilde{K}(y) \right) \left[ \left( \frac{4(LR)^2 - \rho^2}{\rho^2 + 4(LR)^2} \check{e}_\rho - \frac{4LR\rho}{\rho^2 + 4(LR)^2} \check{e}_3 \right) \rho d\varphi - \check{e}_\varphi d\rho \right] \\
\vec{H} &= H_0 \tilde{H}(y) \left( \frac{4LR\rho}{\rho^2 + 4(LR)^2} \check{e}_\rho + \frac{4(LR)^2 - \rho^2}{\rho^2 + 4(LR)^2} \check{e}_3 \right)
\end{aligned} \tag{9}$$

and the equations of motion for the Einstein-Yang Mills-Higgs system take the form

$$\begin{aligned}
\left( y \tilde{f}(y) \right)' &= \frac{1}{R^2} + 3y^2 - \kappa^2 H_0^2 \left( \tilde{f}(y) \tilde{V}_1 + \tilde{V}_2 + \frac{y^2}{2} \frac{\tilde{J}'(y)^2}{\tilde{A}(y)^2} + \frac{\tilde{J}(y)^2 \tilde{K}(y)^2}{\tilde{f}(y) \tilde{A}(y)^2} \right) \\
y \tilde{A}'(y) &= \kappa^2 H_0^2 \left( \tilde{V}_1 + \frac{\tilde{J}(y)^2 \tilde{K}(y)^2}{\tilde{f}(y)^2 \tilde{A}(y)^2} \right) \tilde{A}(y) \\
\left( \tilde{f}(y) \tilde{A}(y) \tilde{K}'(y) \right)' &= \tilde{A}(y) \tilde{K}(y) \left( \frac{\tilde{K}(y)^2 - \frac{1}{R^2}}{y^2} + \frac{1}{\gamma_0^2} \tilde{H}(y)^2 - \frac{1}{\gamma_0^2} \frac{\tilde{J}(y)^2}{\tilde{f}(y) \tilde{A}(y)^2} \right) \\
\left( y^2 \tilde{f}(y) \tilde{A}(y) \tilde{H}'(y) \right)' &= 2\tilde{A}(y) \tilde{H}(y) \tilde{K}(y)^2 \\
\tilde{f}(y) \tilde{A}(y) \left( \frac{y^2 \tilde{J}'(y)}{\tilde{A}(y)} \right)' &= 2 \tilde{J}(y) \tilde{K}(y)^2
\end{aligned} \tag{10}$$

where

$$\tilde{V}_1 = \gamma_0^2 \tilde{K}'(y)^2 + \frac{y^2}{2} \tilde{H}'(y)^2, \quad \tilde{V}_2 = \gamma_0^2 \frac{(\tilde{K}(y)^2 - \frac{1}{R^2})^2}{2 y^2}$$

Here we have defined the dimensionless parameter

$$\gamma_0^2 \equiv -\frac{\Lambda}{3e^2 H_0^2} = \frac{1}{L^2 e^2 H_0^2}, \quad L^2 \equiv -\frac{3}{\Lambda} \tag{11}$$

In the  $R \rightarrow \infty$  limit ansatz (9) becomes

$$\begin{aligned} ds^2 &= -\tilde{f}(y) \tilde{A}(y)^2 d\tau^2 + L^2 \frac{dy^2}{\tilde{f}(y)} + y^2 (dx^{12} + dx^{22}) \\ \vec{A} &= d\tau e H_0 \tilde{J}(y) \check{e}_3 + \frac{\tilde{K}(y)}{L} (\check{e}_1 dx^1 - \check{e}_2 dx^2) \\ \vec{H} &= H_0 \tilde{H}(y) \check{e}_3 \end{aligned} \quad (12)$$

Then system (10) reduces to

$$\begin{aligned} \left( y \tilde{f}(y) \right)' &= 3y^2 - \kappa^2 H_0^2 \left( \tilde{f}(y) \tilde{V}_1 + \tilde{V}_2 + \frac{y^2}{2} \frac{\tilde{J}'(y)^2}{\tilde{A}(y)^2} + \frac{\tilde{J}(y)^2 \tilde{K}(y)^2}{\tilde{f}(y) \tilde{A}(y)^2} \right) \\ y \tilde{A}'(y) &= \kappa^2 H_0^2 \left( \tilde{V}_1 + \frac{\tilde{J}(y)^2 \tilde{K}(y)^2}{\tilde{f}(y)^2 \tilde{A}(y)^2} \right) \tilde{A}(y) \\ \left( \tilde{f}(y) \tilde{A}(y) \tilde{K}'(y) \right)' &= \tilde{A}(y) \tilde{K}(y) \left( \frac{\tilde{K}(y)^2}{y^2} + \frac{1}{\gamma_0^2} \tilde{H}(y)^2 - \frac{1}{\gamma_0^2} \frac{\tilde{J}(y)^2}{\tilde{f}(y) \tilde{A}(y)^2} \right) \\ \left( y^2 \tilde{f}(y) \tilde{A}(y) \tilde{H}'(y) \right)' &= 2\tilde{A}(y) \tilde{H}(y) \tilde{K}(y)^2 \\ \tilde{f}(y) \tilde{A}(y) \left( \frac{y^2 \tilde{J}'(y)}{\tilde{A}(y)} \right)' &= 2 \tilde{J}(y) \tilde{K}(y)^2 \end{aligned} \quad (13)$$

where now

$$\tilde{V}_2 = \gamma_0^2 \frac{\tilde{K}(y)^4}{2 y^2} \quad (14)$$

### 3 The black hole-dyon solution

In reference [11] we constructed self-gravitating dyon solution in asymptotically  $AdS_4$  space which were regular both at the origin and infinity. An important feature noted in that work was that as the dimensionless parameter  $\kappa H_0$  grows, the metric function  $\tilde{f}$  develops a minimum which, at a critical value  $(\kappa H_0)^c$ , corresponds to a zero of  $\tilde{f}$ . For values of  $\kappa H_0 \geq (\kappa H_0)^c$  dyon solution which are regular at the origin ceases to exist; however, as explained in [12]-[13] one can hope that solutions that are finite at spacial infinite and

at one horizon can be found and this will be precisely the issue we shall analyze in this section. We shall construct here such kind of black holes dyon solutions for the Yang-Mills-Higgs system in asymptotically  $AdS_4$  showing that they have finite mass, non-zero horizon radius and, consequently, an associated Hawking temperature so that one can study finite temperature effects in the quantum field theory defined on the boundary.

### 3.1 Boundary conditions and properties of the solution

We shall look for a solution to (10) with a horizon at nonzero radius  $y = y_h$  that we can chose, without loss of generality, to be  $y_h = 1$ . At the horizon we impose the conditions:

$$\tilde{f}(1) = 0 \quad \text{and} \quad A(y), \tilde{K}(y), \tilde{H}(y), \tilde{J}(y)/(y-1) \quad \text{regular at } y=1 \quad (15)$$

so that

$$\begin{aligned} \tilde{f}(y) &= f_1(y-1) + \mathcal{O}[(y-1)^2] \\ \tilde{A}(y) &= a_0 + a_1(y-1) + \mathcal{O}[(y-1)^2] \\ \tilde{H}(y) &= h_0 + h_1(y-1) + \mathcal{O}[(y-1)^2] \\ \tilde{K}(y) &= k_0 + k_1(y-1) + \mathcal{O}[(y-1)^2] \\ \tilde{J}(y) &= j_1(y-1) + \mathcal{O}[(y-1)^2] \end{aligned} \quad (16)$$

Concerning  $y \rightarrow \infty$ , the asymptotic behavior should be

$$\begin{aligned} \tilde{f}(y) &= y^2 + F_0 + \frac{F_1}{y} + \dots \\ \tilde{A}(y) &= 1 + \dots \\ |\tilde{H}(y)| &= H_0 - \frac{H_1}{y^3} + \frac{H_2}{y^5} + \frac{H_3}{y^{4+2\nu}} + \dots, \quad \nu \in \mathbb{R} \end{aligned} \quad (17)$$

$$\tilde{K}(y) = \frac{K_1}{y^{\nu+1}} + \frac{K_3}{y^{\nu+3}}, + \dots \quad (18)$$

$$\tilde{J}(y) = J_0 + \frac{J_1}{y} + \dots \quad (19)$$

Such behavior is consistent with the  $\tilde{K}$  equation of motion whenever the following condition holds

$$\frac{1}{\gamma_0^2} = \nu(\nu+1) \quad (20)$$

Equation (20) has two solutions,

$$\nu_{\pm} = -\frac{1}{2} \pm \frac{1}{2} \sqrt{1 + 4m_W^2 L^2} \quad (21)$$

where  $m_W = eH_0$  is the gauge boson mass. Only the  $\nu_+$  root gives an acceptable asymptotic behavior for  $\tilde{K}(x)$  so that one ends with

$$\tilde{K}(x) = \frac{K_1}{y^{\nu_++1}} = \frac{K_1}{y^{\frac{1}{2}(1+\sqrt{1+4m_W^2 L^2})}} \quad \text{as } x \rightarrow \infty \quad (22)$$

As in [9] we define the field strength associated with the surviving  $U(1)$  symmetry in the form

$$\mathcal{F}_{\mu\nu}^{U(1)} \equiv \frac{H^a}{H_0} F_{\mu\nu}^a \quad (23)$$

with the magnetic and electric fields, and the corresponding charges, given by

$$B^i = \frac{1}{2} \frac{\varepsilon^{ijk}}{\sqrt{g^{(3)}}} \mathcal{F}_{jk}^{U(1)}, \quad E^i = \mathcal{F}^{U(1)i0} \quad (24)$$

$$Q_m = \int d^3x \sqrt{g^{(3)}} \nabla_i^{(3)} B^i = -\frac{4\pi}{e} \quad (25)$$

$$Q_e = \int d^3x \sqrt{g^{(3)}} \nabla_i^{(3)} E^i \quad (26)$$

Note that the spherically symmetric solution has a quantized  $n = -1$  magnetic charge (in units of  $4\pi/e$ ).

Concerning the black hole temperature, it is given by

$$\mathcal{T} = \frac{1}{4\pi L} \tilde{A}(y_h) \tilde{f}'(y_h). \quad (27)$$

Now, taking into account that we have  $y_h = 1$  and defining the dimensionless quantity

$$T \equiv \frac{\mathcal{T}}{eH_0} \quad (28)$$

one has

$$T = \frac{\gamma_0}{4\pi} a_0 f_1 \quad (29)$$

where  $a_0$  and  $f_1$  are defined in (16). Since  $a_0$  and  $f_1$  depend both on  $\kappa$  and  $\gamma_0$  we have that  $T = T(\kappa, \gamma_0)$ . Note that  $T(0, \gamma_0) = (3/4\pi)\gamma_0$ .

### 3.2 Numerical analysis of the solution

We have solved system (10) with boundary conditions (15)-(19) using the relaxation method. In this method the differential equations are replaced by a system of finite-difference equations on a one-dimensional grid with  $N$  points (typically  $N=500$  in our case). Then, the solution is determined by starting from an initial guess and improving it iteratively (see ref.[26] for details).

The system (10) is composed by two first-order differential equations (for the metric functions  $\tilde{f}(y)$  and  $\tilde{A}(y)$ ) and three second-order ones (for the matter fields  $\tilde{K}(y)$ ,  $\tilde{J}(y)$ , and  $\tilde{H}(y)$ ), so to implement the integration algorithm we need eight boundary conditions. At the horizon,  $y_h = 1$ , we choose

$$\begin{aligned} \tilde{f}(1) &= 0, \\ \tilde{J}(1) &= 0, \\ \tilde{f}'(1) \tilde{A}(1) \tilde{K}'(1) &= \tilde{A}(1) \tilde{K}(1) \left( \tilde{K}(1)^2 - \frac{1}{R^2} + \frac{1}{\gamma_0^2} \tilde{H}(1)^2 \right), \\ \tilde{f}'(1) \tilde{A}(1) \tilde{H}'(1) &= 2\tilde{A}(1) \tilde{H}(1) \tilde{K}(1)^2 \end{aligned} \quad (30)$$

The last two conditions are derived by simply evaluating the differential equations for  $\tilde{K}$  and  $\tilde{H}$  at the horizon.

For the boundary corresponding to  $y \rightarrow \infty$  we choose for the numerical calculation a finite size  $L = 10$ , and check that the solutions are stable under variations of  $L$ . We impose at this boundary the conditions

$$\begin{aligned} \tilde{A} &= 1, \\ \tilde{H} &= 1, \\ \tilde{K} &\propto y^{-\nu-1}, \\ \tilde{J} &= J_0 \end{aligned} \quad (31)$$

Following this procedure, we have searched for regular solutions for different values of parameters  $\kappa$  and  $\gamma_0$ . The corresponding solution depends on just one boundary condition, namely  $J_0$ .

As is standard in the relaxation method, its effectiveness is primarily determined by the election of the initial trial functions, especially in this system that comprise eight first-order differential equations. We started by solving the system in the region of small  $\gamma_0$  and  $\kappa$  where the algorithm is

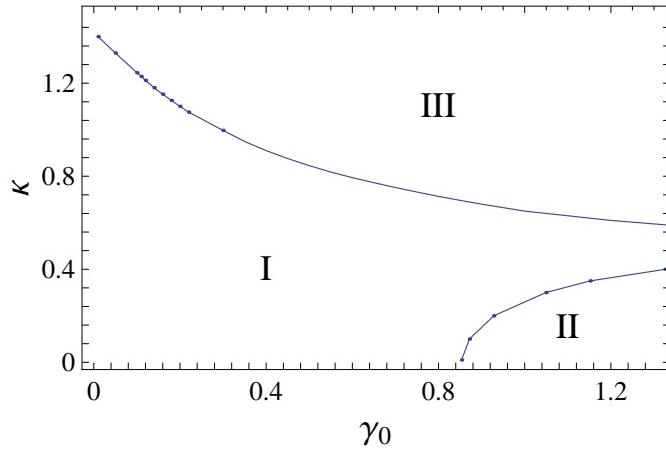


Figure 1: The three regions in parameter space. Only in region I the non trivial self-gravitating dyon solution exists. Region II corresponds to the “trivial” solution and no solution exists in region III whose frontier indicates the appearance of a second horizon.

very stable and the election of trial functions is simple. Once we have found solutions in this region, we select them as trial functions for the equations with larger constants. As we will discuss below, for certain critical values of the parameters there are no non-trivial solutions to the differential system. Near these regions the algorithm becomes unstable and more mesh points and a careful election of the trial functions is needed in order to obtain solutions with the desired accuracy.

Our analysis allows to identify, three regions in parameter space  $(\kappa, \gamma_o)$  with a completely different behavior, as depicted in figure 1. Region I is the one where the gauge and Higgs field solution correspond to a dyon solution, i.e. a soliton with both electric and magnetic charge. Concerning the metric, it corresponds to an asymptotically AdS dyonic black hole. We shall discuss in detail this solution at the end of this section.

In region II, on the right, only a “trivial” solution exists for which the gauge and Higgs fields take the form

$$\tilde{H}(y) = 1, \quad \tilde{K}(y) = 0, \quad \tilde{J} = J_0(1 - \frac{1}{y}) \quad (32)$$

Concerning the metric functions, they correspond to an AdS dyonic black

hole, the result of the back reaction on the geometry

$$\tilde{A}(y) = 1, \quad \tilde{f}(y) = \frac{1}{R^2} + y^2 - \frac{M}{y} + \frac{M - 1 - 1/R^2}{y^2} \quad (33)$$

Here  $M$

$$M = 1 + \frac{1}{R^2} + \frac{\kappa^2 H_0^2}{2} \left( J_0^2 + \frac{\gamma_0^2}{R^4} \right) \quad (34)$$

The reason for the existence of region II is well known: a too large Newton constant destabilizes the dyon solution (its mass grows and its radius decreases). This effect is enhanced in the presence of a cosmological constant so that both mechanisms lead to the occurrence of a critical line after which the dyon becomes gravitationally unstable and only the trivial solution exists (see [9] and references therein for a more detailed discussion).

Finally there exists a region III on the top of the figure, which is bounded from below by a critical line indicating the appearance of a second horizon. No solution, besides the trivial one, exists above this line. For a more detailed discussion on this issues see [12]-[13]).

We have analyzed the solutions both for boundaries  $\partial\mathbf{M} = \mathbf{S}^2 \times \mathbf{S}^1$  (fixed  $R$ ) and for  $\partial\mathbf{M} = \mathbf{R}^2 \times \mathbf{S}^1$  ( $\lim R \rightarrow \infty$ ). Qualitatively the solutions are very similar, thus for definiteness, we shall present in detail the latter case.

Figures 2 shows a representative dyon solution arising in region I. One can see that the Higgs fields rapidly attains its symmetry breaking constant value while the magnetic and electric fields (associated to  $K$  and  $J$  respectively) concentrate in a spherical shell starting at the horizon.

Concerning the metric function  $\tilde{f}$ , one can see that as one increases  $\kappa$  an outer minimum starts to develop (see figure 3) and moves downward, finally reaching zero and becoming an extremal horizon at some critical value  $\kappa_{cr}$  a phenomenon already found for gravitational monopoles in the case  $\Lambda = 0$  [12]-[13]. This defines the critical line between regions I and III. The coefficient  $f_1$ , the slope of  $\tilde{f}$  at the horizon has only a small variation with  $\kappa$ , however the coefficient  $a_0$  that represents the function  $\tilde{A}$  at the horizon, goes from 1, in the absence of back reaction, to 0 at  $\kappa = \kappa_c$ .

We conclude by noting that for small  $\kappa$  the results described above for the solution in region II are qualitatively similar to those obtained in [9] the case in which back reaction is ignored and a Schwarzschild-AdS<sub>4</sub> black hole is

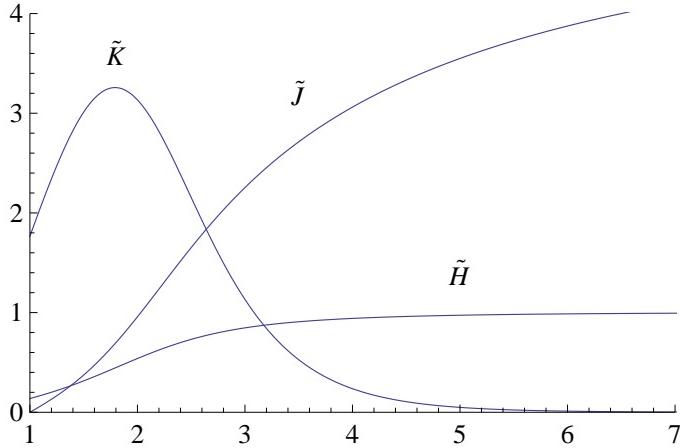


Figure 2: Solution for the gauge field functions ( $\tilde{K}$ ,  $\tilde{J}$  and scalar field  $\tilde{H}$ ) for  $\gamma_0 = 0.1$ ,  $\kappa H_0 = 0.5$  and  $J_0 = 5.5$ . The solution exists starting at the horizon  $y = y_h = 1$ .

taken as a background. This can be explained if one assumes the existence of a weak gravity limit so that the  $\kappa$  dependence is smooth and the solutions change continuously. However, even in this situation similarities are lost for certain ranges of parameters  $(\kappa, \gamma_0)$ , as the phase diagram presented in Fig.1 exhibits a critical line that signals the existence of a second horizon, above which there are no non-trivial solutions. One can associate a Hawking temperature to this second horizon but since the bulk to be considered within the gauge/gravity duality would correspond just to the exterior of this second horizon, where only a trivial solution exists, no relevant physics could arise in this case.

### 3.3 Holographic correspondence at finite temperature

As explained in section 3.1, the Hawking temperature, as given in eqs.(27)-(29), depends on the values of parameters  $\kappa$  and  $\gamma_0$ ,  $T = T(\kappa, \gamma_0)$ . Once these parameters are chosen, the coefficients  $a_0$  and  $f_1$  of the metric functions expansion near the horizon can be calculated and then  $T$  determined from eq.(29).

According to the AdS/CFT correspondence [1] -[3], properties of the dual 3 dimensional field theory defined on the boundary can be read from the

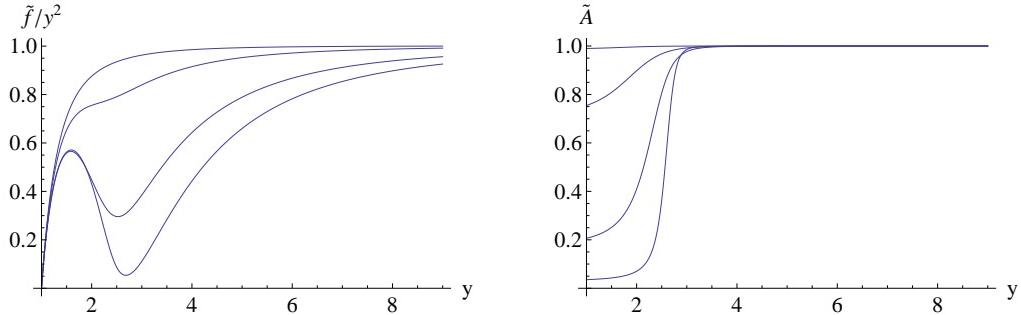


Figure 3: Solution for the metric functions  $\tilde{f}$  and  $\tilde{A}$  as  $\kappa H_0$  grows from  $\kappa = 0.1$  (highest curve) to  $\kappa H_0 = 1.2$  (lowest curve) with  $\gamma_0 = 0.1$  and  $\tilde{J}_0 = 5.5$ . For this case  $\kappa_{cr} = 1.24$ .

asymptotic behavior of the solution of the system in the bulk. In this approach the Hawking temperature  $T(\kappa, \gamma_0)$  corresponds to the temperature of the  $d = 3$  system on the boundary.

Let us first consider the vacuum expectation value in the  $d = 3$  field theory for the operator  $\mathcal{O}_K$ , dual to the function  $K$ , associated with the magnetic field on the bulk. It follows from the identification  $\langle \mathcal{O}_K \rangle \sim K_1$  with  $K_1$  defined in eq.(18) that  $K_1 = K_1(T)$  can be taken as an order parameter for the system in the border. As discussed for different models [4]-[9] one can interpret this result by stating that a condensate is formed above a black hole horizon because of a balance of gravitational and electrostatic forces. One can from eqs.(21)-(22) see that the dimension  $\Delta[\mathcal{O}_K]$  of the order operator is given by [16]

$$\Delta[\mathcal{O}_K] = \frac{3}{2} + \frac{1}{2}\sqrt{1 + 4m_W^2 L^2} = 2 + \nu_+ \quad (35)$$

From the solution that we have found numerically we conclude that a finite temperature continuous symmetry breaking transition takes place so that the system condenses at a critical temperature  $T_c$ , as can be seen from the behavior of  $K_1(T)$  for  $T \approx T_c$  in figure 4.

By fitting the curve we see that near  $T_c$  one has a typical second order phase transition with power behavior of the form

$$\langle \mathcal{O}_K \rangle \sim K_1 \propto (T - T_c)^{1/2} \quad \text{as } T \rightarrow T_c \quad (36)$$

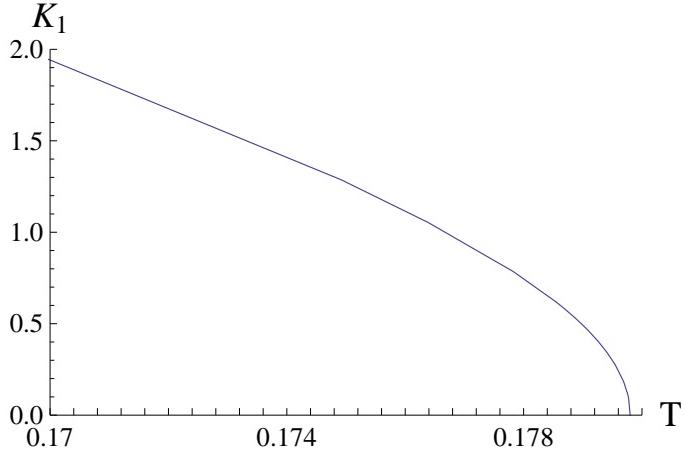


Figure 4: The  $K_1$  coefficient in the gauge field  $A_i$  asymptotic expansion as a function of temperature for  $J_0 = 5.5$  and  $\kappa H_0 = 0.2$ . To the right of the critical temperature (uncondensed phase) the solution corresponds to the trivial one,  $K_1 = 0$ . The critical temperature for this choice of parameters is  $T_c = 0.1798$ .

The free energy  $\mathcal{F}$  associated to the dyon solution can be calculated from the the on-shell Euclidean action action,

$$\mathcal{F} = TS_E|_{\text{on shell}} \quad (37)$$

We have numerically computed  $\mathcal{F}$  for the dyon solution and compared it with the free energy in the uncondensed phase which corresponds to the trivial solution (32). We plot in figure 5 the free energy density difference  $\Delta\mathcal{F}$ , which can be seen to be continuous at  $T = T_c$ .

Other quantities of physical interest like the superconducting charge density can be also computed following the same approach. The main conclusion we can draw form the calculations is that, concerning regions I and II in parameter space, the results obtained by taking into account the back reaction of the matter on the geometry are similar to those obtained in ref. [9] where the Yang-Mills-Higgs system is studied in an AdS black hole background.

On the other hand, the main difference with the fixed background case is that here we find a region in parameter space where a second horizon

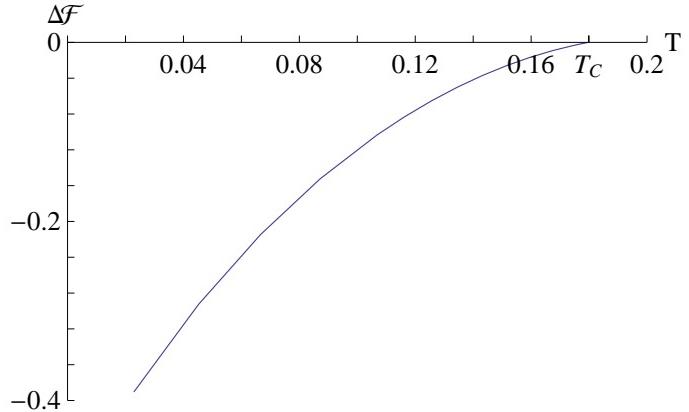


Figure 5: The free energy difference between the condensed and the uncondensed phases. The figure corresponds to  $\kappa H_0 = 0.2$ . To the left of the corresponding critical temperature  $\mathcal{F}_{\text{dyon}} < \mathcal{F}_{\text{trivial}}$ .

arises and no non-trivial solution exists. This implies that one cannot arbitrarily increase Newton's constant and the cosmological constant and have a nontrivial solution leading to a phase transition on the border.

It is important to mention that in the gauge-gravity duality the extra dimension  $r$  does not describe an additional dimension but parameterizes the energy scale in the  $2 + 1$  dual field theory. So, the results described above indicate that the theory in the boundary corresponds to a system undergoing a superconductor phase transition as shown by the non-trivial order parameter that reflects the existence of a condensate.

Let us recall at this point that, as explained before, the  $SU(2)$  gauge invariance of the Lagrangian in the bulk is spontaneously broken to a  $U(1)$  local symmetry due to the non-trivial condition one imposes to the Higgs field at infinity. Such gauge symmetry in the bulk corresponds to a  $U(1)$  global symmetry in the dual field theory, broken by the condensate which also breaks the rotation symmetry in the spatial plane. A combination of the two, the diagonal subgroup, is preserved. Concerning the fate of the corresponding Goldstone boson, in a superconductivity context one should start by coupling the theory on the boundary to a dynamical photon. It can be seen that the Goldstone boson is eaten by the photon, which becomes massive [7].

As it is well known it is the  $SU(2) \rightarrow U(1)$  symmetry breaking pattern in a gauge theory coupled to a Higgs field that allows the existence of regular monopoles and electrically and magnetically charged dyons. One can think that the further breaking of the local  $U(1)$  to the global  $U(1)$  in the boundary could be at the origin of the superfluidity/superconductivity phase regime in the presence of the dyons. Electric repulsion between dyons and the charged horizon compensates gravitational attraction and this is the reason why the superconducting layer floats above the horizon.

Let us end this section by mentioning that it is possible to perform a simple analytic treatment leading to qualitative results in agreement with the detailed numerical analysis discussed above. The method starts by changing the radial coordinate to  $z = r_h/r$  so that the horizon is located at  $z = 1$  and the boundary at infinity at  $z = 0$ . One then proceeds to expand functions  $H$ ,  $K$  and  $J$  near  $z = 1$  and near  $z = 0$  and matching them at  $z = 1/2$  by imposing continuity and smoothness. Using this method, originally proposed in this context in [27] for a simpler model, one can determine the coefficients of both expansions and compute relevant physical quantities. An analysis of this kind applied for a general family of models is in progress and we can advance that one has a good agreement in the case of the model discussed here.

## 4 The Wilson loop

As stated in the introduction, in low dimensional cases ( $d \leq 3$ ), gauge fields in asymptotically  $\text{AdS}_{d+1}$  spaces can induce on the boundary a field theory with dynamical gauge fields [17]-[20]. On this basis, following the gravity/gauge duality approach [21]-[22], we shall compute the Wilson loop for such gauge fields in  $2 + 1$  dimensions in terms of the Nambu-Goto action for the string world sheet defined from the metric solution found in the precedent section. Since such solution corresponds to an asymptotically AdS black hole, our calculation corresponds to finite temperature [23]-[25].

We start from the metric (9)

$$ds^2 = G_{tt}dt^2 + G_{yy}dy^2 + L^2y^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad (38)$$

where

$$G_{tt} = -A^2(y)\tilde{f}(y), \quad G_{yy} = \frac{1}{\tilde{f}(y)} \quad (39)$$

and consider a string world-sheet parameterized in the static gauge according to

$$t(\tau, \sigma) = \tau, \quad \varphi(\tau, \sigma) = \frac{l\sigma}{L \sin \bar{\theta}}, \quad \theta(\tau, \sigma) = \bar{\theta}, \quad y(\tau, \sigma) = y(\sigma) \quad (40)$$

where  $\bar{\theta}$  is a constant and  $y(\sigma)$  defines the embedding, with  $\sigma \in (-1/2, 1/2)$ . The quark  $q$  and anti-quark  $\bar{q}$  are located at points  $(\varphi = \alpha/2, \theta = \bar{\theta})$  and  $(\varphi = -\alpha/2, \theta = \bar{\theta})$  respectively, separated a distance  $l$  along the “parallel”  $\theta = \bar{\theta}$ .

The resulting induced metric  $h_{\alpha\beta}(\sigma) \equiv G_{MN}(X) \partial_\alpha X^M(\sigma) \partial_\beta X^N(\sigma)$  reads

$$h = G_{tt} (d\sigma^0)^2 + (l^2 y(\sigma)^2 + G_{yy} y'(\sigma)^2) (d\sigma)^2 \quad (41)$$

so that the Nambu-Goto action  $S_{NG} = T_s \int d\sigma \sqrt{-\det h_{\alpha\beta}(\sigma)}$  can be written as

$$S_{NG} = -T_s L \int_{-\frac{l}{2L}}^{+\frac{l}{2L}} dx \sqrt{F(u)^2 + G(u)^2 u'(x)^2} \quad (42)$$

with  $u(x) \equiv y(Lx/l)$ ,  $F(u) \equiv u A(u) \mu(u)^{\frac{1}{2}}$  and  $G(u) \equiv A(u)$ .

A minimal action configuration has energy

$$\tilde{E}(l) = T_s L \left( F(u_m) \frac{l}{L} + 2 \int_{u_m}^{u_\infty} du G(u) \sqrt{1 - \frac{F(u_m)^2}{F(u)^2}} \right) \quad (43)$$

where  $u_m$  is the minimum value taken by  $u$  and the distance  $l$  between the quark and the antiquark is given by

$$l = 2 \int_{u_m}^{\infty} \frac{F(u_m)G(u)}{\sqrt{F(u)^2(F(u)^2 - F(u_m)^2)}} du \quad (44)$$

Energy  $E$  as given by eq.(43) diverges linearly for  $u_\infty \rightarrow \infty$ . However by subtracting the bare mass  $m_q$  of the two quarks, each one represented by long strings extended along the  $u$ -direction and puncturing the boundary  $u = u_\infty$  at  $x = \pm \frac{l}{2L}$ ,

$$2 m_q = 2 T_s L \int_{u_h}^{u_\infty} du G(u) \quad (45)$$

we get a finite binding energy in the form

$$E(l) \equiv \tilde{E}(l) - 2m_q = T_s L \left( F(u_m) \frac{l}{L} - 2K\left(\frac{l}{L}\right) \right)$$

$$K\left(\frac{l}{L}\right) = \int_{u_m}^{u_\infty} du G(u) \left( 1 - \sqrt{1 - \frac{F(u_m)^2}{F(u)^2}} \right) + \int_{u_h}^{u_m} du G(u) \quad (46)$$

where  $u_h = 1$  is the position of the horizon. We can re-write the quark-antiquark potential in the form,

$$\frac{E(l)}{l} = T_s \left( F(u_m) - 2 \frac{K\left(\frac{l}{L}\right)}{\frac{l}{L}} \right) \quad (47)$$

The quark-antiquark binding energy  $E = E(l, T)$  can be calculated numerically by eliminating  $u_m$  between eq.(44) and eq.(46). Before doing this, let us analyze the behavior of each one of these functions in terms of  $u_m$ .

One can see in figure 6 that there exists a maximal distance  $l_{max}$  between the quark-anti-quark pair, a phenomenon already found in the study of the finite temperature Wilson line in the large  $N$  limit of  $U(N)$   $\mathcal{N} = 4$  Super Yang-Mills theory in 4 dimensions [23]-[25].

Concerning  $E = E(u_m)$ , the generic behavior in a wide range of parameters values is the one depicted in figure 7. Note that  $E = 0$  for a value  $u_m^c$  satisfying  $u_m^c > u_m^{max}$  leading to  $l^c < l^{max}$ . Thanks to these relations, valid in the whole parameter range that we have studied, the possibility of multi-valuation of the energy  $E$  as exposed by the figure is avoided: as  $l$  grows from  $l = 0$  and reaches  $l^c$ , the energy of the string configuration is the same as the energy of a pair of free quark-antiquark. From there on the answer given by formula (47) does not correspond anymore to the lowest energy which is in fact the free quark energy  $E = 0$ .

Using eqs. (44)-(47) one can determine the energy  $E$  as a function of the distance  $l$  between the quark and the antiquark. The result is presented in figure 8 where we see that there exists a critical distance  $l^c$  at which the quarks become free so that from there on one has a screening behavior.

We have analyzed the behavior of  $l^c$  as a function of the temperature finding that such distance shortens as the temperature grows, a behavior

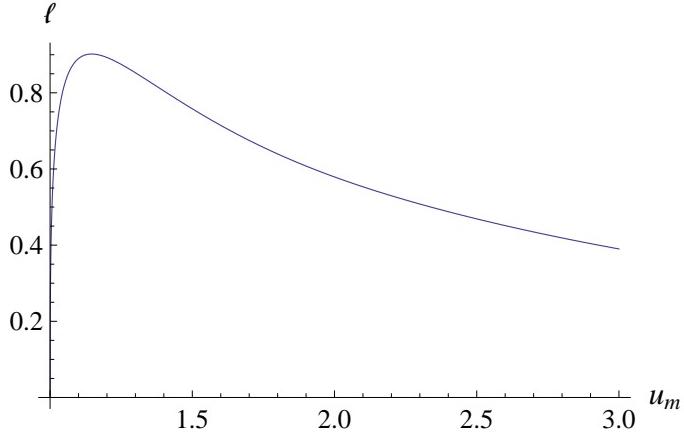


Figure 6: The distance  $l$  between the quark-antiquark pair as a function of  $u_m$ , the minimum value of the variable  $u$ . Note that  $l$  has a maximum possible value,  $l^{max}$ . The curve corresponds to the choice  $\gamma_0 = 0.1$ ,  $\kappa H_0 = 0.5$  and  $J_0 = 5.5$ ; for these parameters  $u_m^{max} = 1.15$  and  $l^{max} = 0.90$  (in units of  $eH_0$ ).

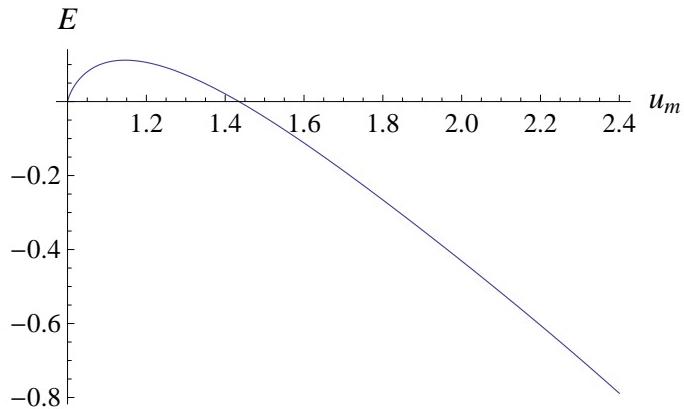


Figure 7: The energy of the quark-antiquark pair relative to the free quark configuration as a function of  $u_m$ , the minimum value of the variable  $u$  ( $\gamma_0 = 0.1$ ,  $\kappa H_0 = 0.5$  and  $J_0 = 5.5$ ). For these parameters one has (in units of  $eH_0$ ,  $u_m^c = 1.44$  and  $l^c = 0.79$ ).

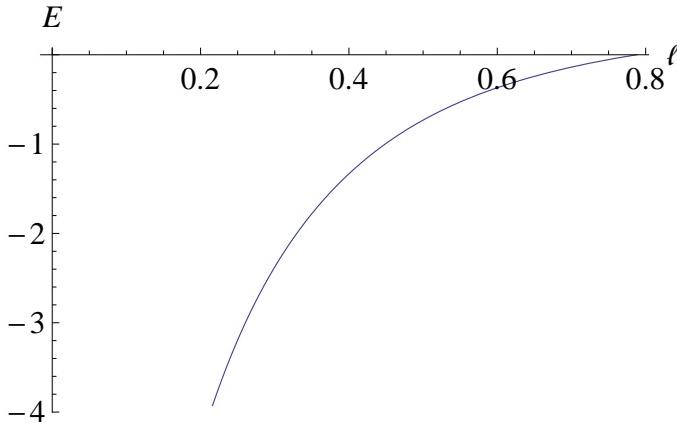


Figure 8: The energy of the quark-antiquark pair as a function of  $l$  ( $\gamma_0 = 0.1$ ,  $\kappa H_0 = 0.5$  and  $J_0 = 5.5$ ). The graph corresponds to a temperature  $T = 0.018$ .

that can be taken as a signal that the screening phenomenon is produced by the thermal bath. A representative graph is given in figure 9.

## 5 Summary and discussion

One of the purposes of the present work was to study the  $d = 3 + 1$  dimensional Einstein-Yang-Mills-Higgs model and find classical solutions to its equations of motion in asymptotically AdS space. As it happens for asymptotically flat space [12]-[13] we have shown that one can construct topologically stable self-gravitating dyons with a metric resulting from the back reaction on the space-time geometry solution that corresponds to a black hole in asymptotically AdS space. We find such solutions in a certain domain of parameter space which corresponds to region I in figure 1. There is also a region where only the trivial solution, for which the Higgs field takes its v.e.v. in the whole space, the gauge field is a pure gauge and the metric corresponds to a Schwarzschild-AdS black hole (region II). As for the behavior of the Higgs and gauge fields spherically symmetric solutions, they are very similar as those we presented in ref. [9] for the case of a Schwarzschild-AdS black hole background (no back reaction), with the horizon radius inside the

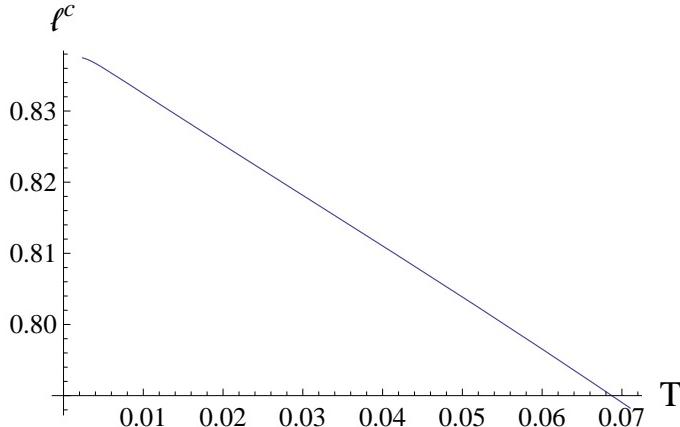


Figure 9: The distance  $l^c$  at which the quark and the anti-quark become free as a function of temperature. ( $\gamma_0 = 0.3$ ).

dyon core and the Higgs field rapidly taking its vacuum expectation value. Finally there is a critical line which marks the appearance of a double horizon solution like the Reissner-Nordstrom metric. Above such line, no solution exists.

Using the gauge-gravity correspondence, we were able to study properties of the dual  $d = 2 + 1$  quantum field theory on the boundary of the  $3 + 1$  asymptotical AdS space where our classical solutions were found. Because those solutions correspond to a black hole metric, they have an associated Hawking temperature  $T = T(\gamma_0, \kappa)$  with parameters  $\gamma_0$  and  $\kappa$  related to the cosmological and Newton constants respectively. Then, varying those parameters in the bulk classical solutions we were able to study the finite temperature behavior of the vacuum expectation values of operators defined on the boundary, identifying an order parameter and showing that a second order phase transition takes place.

Finally, because we have taken into account the back reaction of matter on geometry we were able to use the resulting black hole metric to calculate the Nambu-Goto action for our black hole solution related through the holographic correspondence to the Wilson loop at finite temperature. In this way we were able to evaluate the binding energy  $E = E(l, T)$  of two external charges in the boundary (a quark-antiquark pair). We found a inverse power law behavior for  $E$  as a function of the distance between quarks, up to

a maximal distance  $l_c$  where  $E$  vanishes indicating that the quarks become free in a typical screening process which can be attributed to the thermal bath. It should be remarked that this scenario is qualitatively the same already found on the pioneering works on finite temperature Wilson loops in the large  $N$  limit of  $U(N)$   $\mathcal{N} = 4$  supersymmetric Yang-Mills theory in 4 dimensions [23]-[25].

Let us note at this point that, following the approach applied here, one can establish the existence of confinement for Yang-Mills theory in three dimensions by computing Wilson loops at finite temperature using the string/gauge correspondence (see ([28]) for a detailed calculation). We have found, in contrast, a screening behavior so that we can discard the possibility that the model in the boundary could be related directly to  $2 + 1$  Yang-Mills. It is however well known that when massive fermions are added to a  $2 + 1$  gauge theory the effective action resulting from integration of fermions induces a Chern-Simons term and, as a result, confinement is destroyed (see for example [29]). Then, one can speculate that the theory on the boundary could include a Chern-Simons term. This idea may be tested by going beyond the classical approximation on the gravity side which would require to embed the gravity model into string theory. We hope to present a thorough study on these issues in the future.

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